ON WORK-HARDENING OF PLASTIC SOLIDS

(OB UPROCHNENII ZHESTKO-PLASTICHESKOGO MATERIALA)

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KHUAN KE-CHZHI
(Moscow, Pekin)

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A law of work-hardening of plastic solids has been offered in a paper [1] by Prager. According to this law the yield surface moves translationally in the stress space. Budiansky [2] noted that, if applied to two-dimensional state of stress, this law contradicts the assumption of original isotropy of the material. In the present note the law of deformation for the plane state of stress is treated as a special case of the law for the three-dimensional state of stress.

Denote by σ_1, σ_2, σ_3 the stress components and by e_1, e_2, e_3 the components of deformation. Tresca's condition of initial yield is represented by a hexagonal prism in the stress space and is expressed by the formula

$$Q = \max_{1 \leqslant i \leqslant 6} f_i(\sigma_1, \sigma_2, \sigma_3) = f_{i*}(\sigma_1, \sigma_2, \sigma_3)$$

where

$$f_1 = \sigma_1 - \sigma_3, \quad f_2 = \sigma_2 - \sigma_3, \quad f_3 = \sigma_2 - \sigma_1,$$
$$f_4 = \sigma_3 - \sigma_1, \quad f_5 = \sigma_3 - \sigma_2, \quad f_6 = \sigma_1 - \sigma_2$$

The quantity Q characterizes the dimensions of the prism, while i^* indicates the plane in which the stress point appears. In the process of hardening, which is assumed to be linear, the hexagonal prism changes, in a manner preserving similarity, with a moving axis of similarity. Denote by c' Prager's modulus of linear hardening and by c'' the modulus of linear isotropic hardening. Then the components of displacement of the axis of the prism will be $c'e_1$, $c'e_2$, $c'e_3$. Denoting by a dot differentiation with respect to time, the law of deformation assumes the form

$$\text{(a)} \qquad c''\dot{e}_\alpha = \frac{\partial f_i}{\partial \sigma_\alpha} \dot{Q} \qquad (\alpha = 1, 2, 3)$$

$$Q = f_i(\sigma_1 - c'e_1, \sigma_2 - c'e_2, \sigma_3 - c'e_3) \qquad (1)$$

if the stress point remains in the plane i, or

$$c''\dot{e}_\alpha = \left(\mu \frac{\partial f_i}{\partial \sigma_\alpha} + \nu \frac{\partial f_j}{\partial \sigma_\alpha}\right)\dot{Q} \qquad (\alpha=1,2,3,\ \mu+\nu=1)$$

(б) $$Q = f_i(\sigma_1 - c'e_1, \sigma_2 - c'e_2, \sigma_3 - c'e_3) = f_j(\sigma_1 - c'e_1, \sigma_2 - c'e_2, \sigma_3 - c'e_3) \qquad (2)$$

if the stress point appears on the edge, representing the intersection of the planes i and j. From the equations (1) and (2) we find

(a) $$e_\alpha = \frac{1}{d}\frac{\partial f_i}{\partial \sigma_\alpha}\dot{f}_i \quad (\alpha=1,2,3), \qquad \dot{Q} = \lambda \dot{f}_i \qquad (3)$$

(б) $$\dot{e}_\alpha = \frac{4}{d(1-\lambda)(3+\lambda)}\left[\frac{\partial f_i}{\partial \sigma_\alpha}\dot{f}_i + \frac{\partial f_j}{\partial \sigma_\alpha}\dot{f}_j - \frac{1+\lambda}{2}\left(\frac{\partial f_i}{\partial \sigma_\alpha}\dot{f}_j + \frac{\partial f_j}{\partial \sigma_\alpha}\dot{f}_i\right)\right]$$

$$\dot{Q} = \frac{2\lambda}{3+\lambda}[\dot{f}_i + \dot{f}_j] \qquad (4)$$

where $$\dot{f}_i = \dot{f}_i(\sigma_1, \sigma_2, \sigma_3), \qquad d = 2c' + c'', \qquad \lambda = \frac{c''}{d} \qquad (0 \leqslant \lambda \leqslant 1)$$

The case $\lambda = 0$ corresponds Prager's law of hardening, while $\lambda = 1$ characterizes isotropic hardening.

If at some instant the stress point appears on an edge of the hexagonal prism, then it remains on that edge, if

$$\frac{2\lambda}{3+\lambda}(\dot{f}_i + \dot{f}_j) > \lambda\dot{f}_i, \qquad \frac{2\lambda}{3+\lambda}(\dot{f}_i + \dot{f}_j) > \lambda\dot{f}_j$$

or it goes over to the plane i (or j), when

$$\lambda\dot{f}_i \ (\text{or } \lambda\dot{f}_j) > \frac{2\lambda}{3+\lambda}(\dot{f}_i + \dot{f}_j), \qquad \lambda\dot{f}_i \ (\text{or } \lambda\dot{f}_j) > \lambda\dot{f}_j \ (\text{or } \lambda\dot{f}_i)$$

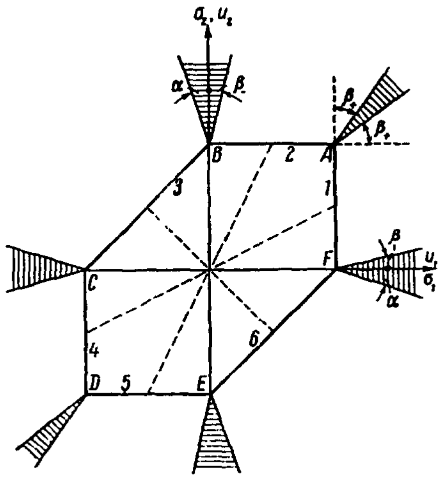

Fig. 1.

Consider the state of plane stress ($\sigma_3 = 0$). Tresca's yield condition in the plane (σ_1, σ_2) can be represented by a hexagon *ABCDEF*, as shown in Fig. 1, where

$$\alpha = \operatorname{arc\,tg}\frac{1-\lambda}{1+\lambda}, \qquad \beta_+ = \operatorname{arc\,tg}\frac{1+\lambda}{2}, \qquad \beta_- = \operatorname{arc\,tg}\frac{1-\lambda}{2}$$

On the basis of the equations (3) and (4), we can find the strain law for the state of plane stress in a form given in the following table:

Regime	Strain law	Speed of Expansion of hexagon
F	$\dot{e}_1 = \frac{4}{d(3+\lambda)}[\dot{\sigma}_1 - \frac{1}{2}\dot{\sigma}_2]$ $\dot{e}_2 = \frac{4}{d(1-\lambda)(3+\lambda)}[\dot{\sigma}_2 - \frac{1-\lambda}{2}\dot{\sigma}_1]$ $\dot{\sigma}_1 = d[\dot{e}_1 + \frac{1-\lambda}{2}\dot{e}_2]$ $\dot{\sigma}_2 = d(1-\lambda)[\dot{e}_2 + \frac{1}{2}\dot{e}_1]$	$\dot{Q} = \frac{2\lambda}{3+\lambda}(2\dot{\sigma}_1 - \dot{\sigma}_2)$
FA	$\dot{e}_1 = \dot{\sigma}_1/d,\ \dot{e}_2 = 0;\ \dot{\sigma}_1 = d\dot{e}_1$	$\dot{Q} = \lambda\dot{\sigma}_1$
A	$\dot{e}_1 = \frac{4}{d(1-\lambda)(3+\lambda)}[\dot{\sigma}_1 - \frac{1+\lambda}{2}\dot{\sigma}_2]$ $\dot{e}_2 = \frac{4}{d(1-\lambda)(3+\lambda)}[\dot{\sigma}_2 - \frac{1+\lambda}{2}\dot{\sigma}_1]$ $\dot{\sigma}_1 = d[\dot{e}_1 + \frac{1+\lambda}{2}\dot{e}_2]$ $\dot{\sigma}_2 = d[\dot{e}_2 + \frac{1+\lambda}{2}\dot{e}_1]$	$\dot{Q} = \frac{2\lambda}{3+\lambda}(\dot{\sigma}_1 + \dot{\sigma}_2)$
AB	$\dot{e}_1 = 0,\ \dot{e}_2 = \frac{\dot{\sigma}_2}{d};\ \dot{\sigma}_2 = d\dot{e}_2$	$\dot{Q} = \lambda\dot{\sigma}_2$
B	$\dot{e}_1 = \frac{4}{d(1-\lambda)(3+\lambda)}[\dot{\sigma}_1 - \frac{1-\lambda}{2}\dot{\sigma}_2]$ $\dot{e}_2 = \frac{4}{d(3+\lambda)}[\dot{\sigma}_2 - \frac{1}{2}\dot{\sigma}_1]$ $\dot{\sigma}_1 = d(1-\lambda)[\dot{e}_1 + \frac{1}{2}\dot{e}_2]$ $\dot{\sigma}_2 = d[\dot{e}_2 + \frac{1-\lambda}{2}\dot{e}_1]$	$\dot{Q} = \frac{2\lambda}{3+\lambda}(2\dot{\sigma}_2 - \dot{\sigma}_1)$
BC	$\dot{e}_1 = \frac{\dot{\sigma}_1 - \dot{\sigma}_2}{d},\ \dot{e}_2 = \frac{\dot{\sigma}_2 - \dot{\sigma}_1}{d}$ $\dot{\sigma}_2 - \dot{\sigma}_1 = d\dot{e}_2 = -d\dot{e}_1$	$\dot{Q} = \lambda(\dot{\sigma}_2 - \dot{\sigma}_1)$

The displacement components u_1, u_2 of the center of the hexagon in the plane (σ_1, σ_2) are connected with the displacement components $c'e_1$, $c'e_2$, $c'e_3$ of the axis of the prism in the space $(\sigma_1\ \sigma_2\ \sigma_3)$ by the formulas

$$u_1 = c'(2e_1 + e_3) = d(1-\lambda)\left(e_1 + \tfrac{1}{2} e_3\right), \quad u_2 = c'(2e_3 + e_1) = d(1-\lambda)\left(e_2 + \tfrac{1}{2} e_1\right) \qquad (5)$$

It follows that if a stress point is located on a side of the hexagon, then the center of the hexagon will move, in the process of hardening, along the line connecting the center with the midpoint of the side. The stress point will be always on the edge if the stress increment vector is located in the wedge-like domain shown in Fig. 1. We give the most simple examples.

Example 1. Simple extension $\sigma_1 = \phi(t)$, $\sigma_2 = \sigma_3 = 0$ $(\phi >)$. The deformation is

$$e_1 = \frac{4}{d(3+\lambda)}(\varphi - \sigma_0), \qquad e_2 = e_3 = -\frac{e_1}{2} \ (\text{regime} F) \qquad (6)$$

where σ_0 is the initial yield limit.

Example 2. Two dimensional compression $\sigma_1 = \sigma_2 = -\phi(t)$, $\sigma_3 = 0$ $(\phi > 0)$. The deformation is

$$e_3 = \frac{4}{d(3+\lambda)}(\varphi - \sigma_0), \qquad e_1 = e_2 = -\frac{e_3}{2} \quad (\text{regime} D) \qquad (7)$$

Comparing the expressions (6) and (7) with each other we see that the contradiction indicated by Budiansky [2] does not exist here.

Example 3. A circular plate, supported along its boundary and acted upon by uniformly distributed pressure p. We give here the results for the cases $\lambda = 0$ and $\lambda = 1$ only. In the case of isotropic hardening we obtain a solution identical with that of hodge [3], who uses α instead of λ. In the case of Prager hardening $(\lambda = 0)$ the solution becomes

$$\frac{cw}{M_0R^2} = -\frac{3}{32}P(1 + 3\rho_1^4 - \rho_2^4 - 8\rho_1^3\rho_2 + 6\rho_1^2\rho^2 - \rho^4) - K_1(1-\rho_2^2) + \left(K_2 - \frac{1}{4}\rho_2^2\right)\ln\rho_2 \qquad \text{when } 0 \leqslant \rho \leqslant \rho_1$$

$$= -\frac{3}{32}P(1 - \rho_2^4 - 8\rho_1^3\rho_2 + 8\rho_1^3\rho) - K_1(1-\rho_2^2) + \left(K_2 - \frac{1}{4}\rho_2^2\right)\ln\rho_2 \qquad \text{when } \rho_1 \leqslant \rho \leqslant \rho_2$$

$$= -\frac{3}{32}P(1-\rho^4) - \frac{1}{4}\rho^2\ln\rho - K_1(1-\rho^2) + K_2\ln\rho \qquad \text{when } \rho_2 \leqslant \rho \leqslant 1$$

$$\frac{M_r}{M_0} = 1 + \frac{3}{16}P(9\rho_1^2 - 7\rho^2) \qquad \text{when } 0 \leqslant \rho \leqslant \rho_1$$

$$= 1 + \frac{1}{8}P\rho^{-1}(11\rho_1^3 - 8\rho^3) + \frac{3}{4}P\rho_1^3\rho^{-1}\ln\frac{\rho}{\rho_1} \qquad \text{when } \rho_1 \leqslant \rho \leqslant \rho_2$$

$$= \frac{7}{8} + \frac{3}{4}\ln\rho - \frac{21}{16}P\rho^2 - 3K_1 + \frac{1}{2}K_2\rho^{-2} \qquad \text{when} \quad \rho_2 \leqslant \rho \leqslant 1$$

$$\frac{M_\theta}{M_0} = 1 + \frac{3}{16}P(9\rho_1^2 - 5\rho^2) \qquad \text{when} \quad 0 \leqslant \rho \leqslant \rho_1$$

$$= 1 + \frac{3}{4}P\rho_1^3\rho^{-1} \qquad \text{when} \quad \rho_1 \leqslant \rho \leqslant \rho_2$$

$$= \frac{13}{8} + \frac{3}{4}\ln\rho - \frac{15}{16}P\rho^2 - 3K_1 - \frac{1}{2}K_2\rho^{-2} \qquad \text{when} \quad \rho_2 \leqslant \rho \leqslant 1$$

where (8)

$$c = \frac{dh^3}{12}, \qquad M_0 = \frac{\sigma_0 h^2}{4}, \qquad P = \frac{p}{p_0} = \frac{pR^2}{6M_0}, \qquad \rho = \frac{r}{R}$$

R = plate radius; h = thickness; w = deflection; M_r, M_θ = bending moments;

$$K_1 = -[6\rho_2^2 - 14 + 4\rho_2^2\ln\rho_2 + 3P(7 - 3\rho_2^4)] / 16(3 - \rho_2^2)$$

$$K_2 = \rho_2^2[3P(9\rho_2^2 - 7) - 4(1 + 3\ln\rho_2)] / 8(3 - \rho_2^2)$$

while ρ_1, ρ_2 are determined by the equations

$$3P\rho_1^3(3\rho_2^{-1} - \rho_2) = 3P(7 + \rho_2^2)(1 - \rho_2^2) - 2(1 - \rho_2^2) + 12\ln\rho_2$$

$$P\rho_1^3\left(11 + 6\ln\frac{\rho_2}{\rho_1}\right)(3\rho_2^{-1} - \rho_2) = P(21 + 6\rho_2^2 - 11\rho_2^4) - 26 + 10\rho_2^2 + 12\ln\rho_2$$

At the beginning of the yield process ($P = 1$) we have $\rho_1 = 0$, $\rho_2 = 1$; when $P \to \infty$, both ρ_1 and ρ_2 tend toward the limit $\sqrt{7/3}$. When $P = 1.5$, we have $\rho_1 = 0.5708$, $\rho_2 = 0.9670$ and the solution (8) nearly coincides with that of Hodge [3] for the intermediate case $a = 0.5$.

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